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Monitoring of induced distributed double-couple sources using Marchenko-based virtual receivers

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S1 Classical homogeneous Green's function representation

S1.1 Definition of the homogeneous Green's function

Consider an inhomogeneous lossless acoustic medium with mass density $\rho(\boldsymbol{x})$ and compressibility $\kappa(\boldsymbol{x})$. In this medium a space- and time-dependent source distribution $q(\boldsymbol{x},t)$ is present, with q defined as the volume-injection rate density. The acoustic wave field, caused by this source distribution, is described in terms of the acoustic pressure $p(\boldsymbol{x},t)$ and the particle velocity $v_i(\boldsymbol{x},t)$. These field quantities obey the equation of motion and the stress-strain relation, according to

$$\rho\partial_t v_i + \partial_i p = 0, \tag{S1}$$
$$\kappa\partial_t p + \partial_i v_i = q. \tag{S2}$$

When q is an impulsive source at $\boldsymbol{x} = \boldsymbol{x}_{\rm A}$ and $t = 0$, according to

$$q(\boldsymbol{x},t) = \delta(\boldsymbol{x}-\boldsymbol{x}_{\rm A})\delta(t), \tag{S3}$$

then the causal solution of Eqs. (S1) and (S2) defines the Green's function, hence

$$p(\boldsymbol{x},t) = G(\boldsymbol{x},\boldsymbol{x}_{\rm A},t). \tag{S4}$$

By eliminating v_i from Eqs. (S1) and (S2) and substituting Eqs. (S3) and (S4), we find that the Green's function $G(\boldsymbol{x},\boldsymbol{x}_{\rm A},t)$ obeys the following wave equation

$$\partial_i(\rho^{-1}\partial_i G) - \kappa\partial_t^2 G = -\delta(\boldsymbol{x}-\boldsymbol{x}_{\rm A})\partial_t\delta(t). \tag{S5}$$

Wave equation (S5) is symmetric in time, except for the source on the right-hand side, which is anti-symmetric. Hence, the time-reversed Green's function $G(\boldsymbol{x},\boldsymbol{x}_{\rm A},-t)$ obeys the same wave equation, but with opposite sign for the source. By summing the wave equations for $G(\boldsymbol{x},\boldsymbol{x}_{\rm A},t)$ and $G(\boldsymbol{x},\boldsymbol{x}_{\rm A},-t)$, the sources on the right-hand sides cancel each other, hence, the homogeneous Green's function

$$G_{\rm h}(\boldsymbol{x},\boldsymbol{x}_{\rm A},t) = G(\boldsymbol{x},\boldsymbol{x}_{\rm A},t) + G(\boldsymbol{x},\boldsymbol{x}_{\rm A},-t) \tag{S6}$$

obeys the homogeneous equation

$$\partial_i(\rho^{-1}\partial_i G_{\rm h}) - \kappa\partial_t^2 G_{\rm h} = 0. \tag{S7}$$

S1.2 Reciprocity theorems

We define the temporal Fourier transform of a time-dependent quantity $u(t)$ as

$$u(\omega) = \int_{-\infty}^{\infty} u(t)\exp(i\omega t){\rm d}t. \tag{S8}$$

In the frequency domain, Eqs. (S1) and (S2) transform to

$$-i\omega\rho v_i + \partial_i p = 0, \tag{S9}$$
$$-i\omega\kappa p + \partial_i v_i = q. \tag{S10}$$

We introduce two independent acoustic states, which will be distinguished by subscripts A and B. Rayleigh's reciprocity theorem is obtained by considering the quantity $\partial_i\{p_{\rm A}v_{i,B} - v_{i,A}p_{\rm B}\}$, applying the product rule for differentiation, substituting Eqs. (S9) and (S10) for both states, integrating the result over a spatial domain $\mathbb{V}$ enclosed by surface $\mathbb{S}$ with outward pointing normal n_i, and applying the theorem of Gauss (de Hoop, 1988; Fokkema and van den Berg, 1993). Assuming that in $\mathbb{V}$ the medium parameters $\rho(\boldsymbol{x})$ and $\kappa(\boldsymbol{x})$ in the two states are identical, this yields Rayleigh's reciprocity theorem of the convolution type

$$\int_{\mathbb{V}} \{p_{\rm A}q_{\rm B} - q_{\rm A}p_{\rm B}\}{\rm d}\boldsymbol{x} = \oint_{\mathbb{S}} \frac{1}{i\omega\rho}\{p_{\rm A}(\partial_i p_{\rm B}) - (\partial_i p_{\rm A})p_{\rm B}\}n_i{\rm d}\boldsymbol{x}. \tag{S11}$$

We derive a second form of Rayleigh's reciprocity theorem for time-reversed wave fields. In the frequency domain, time-reversal is replaced by complex conjugation. When p is a solution of Eqs. (S9) and (S10) with source distribution q (and real-valued medium parameters), then p^* obeys the same equations with source distribution $-q^*$. Making these substitutions for state A in Eq. (S11) we obtain Rayleigh's reciprocity theorem of the correlation type (Bojarski, 1983)

$$\int_{\mathbb{V}} \{p_{\rm A}^*q_{\rm B} + q_{\rm A}^*p_{\rm B}\}{\rm d}\boldsymbol{x} = \oint_{\mathbb{S}} \frac{1}{i\omega\rho}\{p_{\rm A}^*(\partial_i p_{\rm B}) - (\partial_i p_{\rm A}^*)p_{\rm B}\}n_i{\rm d}\boldsymbol{x}. \tag{S12}$$

S1.3 Representation of the homogeneous Green's function

We choose point sources in both states, according to $q_{\rm A}(\boldsymbol{x},\omega) = \delta(\boldsymbol{x}-\boldsymbol{x}_{\rm A})$ and $q_{\rm B}(\boldsymbol{x},\omega) = \delta(\boldsymbol{x}-\boldsymbol{x}_{\rm B})$, with $\boldsymbol{x}_{\rm A}$ and $\boldsymbol{x}_{\rm B}$ both in $\mathbb{V}$. The fields in states A and B are thus expressed in terms of Green's functions, according to

$$p_{\rm A}(\boldsymbol{x},\omega) = G(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega), \tag{S13}$$
$$p_{\rm B}(\boldsymbol{x},\omega) = G(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega), \tag{S14}$$

with $G(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)$ and $G(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)$ being the Fourier transforms of $G(\boldsymbol{x},\boldsymbol{x}_{\rm A},t)$ and $G(\boldsymbol{x},\boldsymbol{x}_{\rm B},t)$, respectively. Making these substitutions in Eq. (S12) and using source-receiver reciprocity of the Green's functions gives (Porter, 1970; Oristaglio, 1989; Wapenaar, 2004; Van Manen et al., 2005)

$$G_{\rm h}(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega) = \oint_{\mathbb{S}} \frac{1}{i\omega\rho(\boldsymbol{x})}\Big(\{\partial_i G(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\}G^*(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega) - G(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\partial_i G^*(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\Big)n_i{\rm d}\boldsymbol{x}, \tag{S15}$$

where $G_{\rm h}(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)$ is the homogeneous Green's function in the frequency domain. It is defined as

$$G_{\rm h}(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega) = G(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega) + G^*(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega) = 2\Re\{G(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\}, \tag{S16}$$

where $\Re$ denotes the real part. Equation (S15) is an exact representation for the homogeneous Green's function $G_{\rm h}(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)$.

When $\mathbb{S}$ is sufficiently smooth and the medium outside $\mathbb{S}$ is homogeneous (with mass density ρ_0, compressibility κ_0 and propagation velocity $c_0 = (\kappa_0\rho_0)^{-1/2}$), the two terms under the integral in Eq. (S15) are nearly identical (but opposite in sign), hence

$$G_{\rm h}(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega) = -2\oint_{\mathbb{S}} \frac{1}{i\omega\rho_0}G(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\partial_i G^*(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)n_i{\rm d}\boldsymbol{x}. \tag{S17}$$

The main approximation is that evanescent waves are neglected at $\mathbb{S}$ (Zheng et al., 2011; Wapenaar et al., 2011).

S2 Single-sided homogeneous Green's function representations

S2.1 Modification of the configuration

We replace the arbitrary closed surface $\mathbb{S}$ by a combination of two surfaces $\mathbb{S}_0$ and $\mathbb{S}_{\rm A}$, as indicated in Fig. S1. Here $\mathbb{S}_0$ may be curved, but $\mathbb{S}_{\rm A}$ is a horizontal surface, with $\boldsymbol{n} = (0,0,1)$. The depth level of $\mathbb{S}_{\rm A}$ is defined as $x_{3,{\rm A}}$ (which is equal to

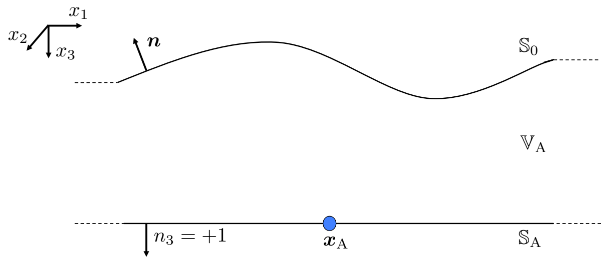

Figure S1. Modified configuration. The surface $\mathbb{S}$ consists of the combination of surfaces $\mathbb{S}_0$ and $\mathbb{S}_\mathrm{A}$.

the x_3-coordinate of the point $\boldsymbol{x}_\mathrm{A}$). The domain between surfaces $\mathbb{S}_0$ and $\mathbb{S}_\mathrm{A}$ is called $\mathbb{V}_\mathrm{A}$. For this configuration, reciprocity theorems (S11) and (S12) are replaced by

$$\int_{\mathbb{V}_\mathrm{A}} \{p_\mathrm{A} q_\mathrm{B} - q_\mathrm{A} p_\mathrm{B}\} \mathrm{d}\boldsymbol{x} = \int_{\mathbb{S}_0} \frac{1}{i\omega\rho} \{p_\mathrm{A}(\partial_i p_\mathrm{B}) - (\partial_i p_\mathrm{A}) p_\mathrm{B}\} n_i \mathrm{d}\boldsymbol{x} + \int_{\mathbb{S}_\mathrm{A}} \frac{1}{i\omega\rho} \{p_\mathrm{A}(\partial_3 p_\mathrm{B}) - (\partial_3 p_\mathrm{A}) p_\mathrm{B}\} \mathrm{d}\boldsymbol{x} \tag{S18}$$

and

$$\int_{\mathbb{V}_\mathrm{A}} \{p_\mathrm{A}^* q_\mathrm{B} + q_\mathrm{A}^* p_\mathrm{B}\} \mathrm{d}\boldsymbol{x} = \int_{\mathbb{S}_0} \frac{1}{i\omega\rho} \{p_\mathrm{A}^*(\partial_i p_\mathrm{B}) - (\partial_i p_\mathrm{A}^*) p_\mathrm{B}\} n_i \mathrm{d}\boldsymbol{x} + \int_{\mathbb{S}_\mathrm{A}} \frac{1}{i\omega\rho} \{p_\mathrm{A}^*(\partial_3 p_\mathrm{B}) - (\partial_3 p_\mathrm{A}^*) p_\mathrm{B}\} \mathrm{d}\boldsymbol{x}, \tag{S19}$$

respectively. In the following we use these reciprocity theorems as the basis for deriving several versions of single-sided homogeneous Green's function representations, each time by applying decomposition to one or more of the integrals in these theorems.

S2.2 Single-sided homogeneous Green's function representation: general formulation

Following a similar derivation as in Appendix B in Wapenaar and Berkhout (1989), we reformulate Eqs. (S18) and (S19) as

$$\int_{\mathbb{V}_\mathrm{A}} (p_\mathrm{A} q_\mathrm{B} - q_\mathrm{A} p_\mathrm{B}) \mathrm{d}\boldsymbol{x} = \int_{\mathbb{S}_0} \frac{1}{i\omega\rho} \Big(p_\mathrm{A}(\partial_i p_\mathrm{B}) - (\partial_i p_\mathrm{A}) p_\mathrm{B}\Big) n_i \mathrm{d}\boldsymbol{x} - \int_{\mathbb{S}_\mathrm{A}} \frac{2}{i\omega\rho} \Big((\partial_3 p_\mathrm{A}^+) p_\mathrm{B}^- + (\partial_3 p_\mathrm{A}^-) p_\mathrm{B}^+\Big) \mathrm{d}\boldsymbol{x} \tag{S20}$$

and, ignoring evanescent waves,

$$\int_{\mathbb{V}_\mathrm{A}} (p_\mathrm{A}^* q_\mathrm{B} + q_\mathrm{A}^* p_\mathrm{B}) \mathrm{d}\boldsymbol{x} = \int_{\mathbb{S}_0} \frac{1}{i\omega\rho} \Big(p_\mathrm{A}^*(\partial_i p_\mathrm{B}) - (\partial_i p_\mathrm{A}^*) p_\mathrm{B}\Big) n_i \mathrm{d}\boldsymbol{x} - \int_{\mathbb{S}_\mathrm{A}} \frac{2}{i\omega\rho} \Big((\partial_3 p_\mathrm{A}^+)^* p_\mathrm{B}^+ + (\partial_3 p_\mathrm{A}^-)^* p_\mathrm{B}^-\Big) \mathrm{d}\boldsymbol{x}. \tag{S21}$$

The superscripts $+$ and $-$ stand for downgoing and upgoing, respectively. For state A we consider the focusing function $f_1(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega) = f_1^+(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega) + f_1^-(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega)$, introduced in section 3.1 in the companion paper "Green's theorem in seismic imaging across the scales" (Wapenaar et al., 2019). This focusing function is defined in a truncated version of the medium, which is identical to the actual medium in $\mathbb{V}_\mathrm{A}$, but reflection free above $\mathbb{S}_0$ and below $\mathbb{S}_\mathrm{A}$. The focusing conditions at the focal plane $\mathbb{S}_\mathrm{A}$ are (Wapenaar et al., 2014)

$$[\partial_3 f_1^+(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega)]_{x_3=x_{3,\mathrm{A}}} = \tfrac{1}{2} i\omega\rho(\boldsymbol{x}_\mathrm{A})\delta(\boldsymbol{x}_\mathrm{H} - \boldsymbol{x}_\mathrm{H,A}), \tag{S22}$$

$$[\partial_3 f_1^-(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega)]_{x_3=x_{3,\mathrm{A}}} = 0. \tag{S23}$$

For state B we consider the Green's function $G(\boldsymbol{x},\boldsymbol{x}_\mathrm{B},\omega) = G^+(\boldsymbol{x},\boldsymbol{x}_\mathrm{B},\omega) + G^-(\boldsymbol{x},\boldsymbol{x}_\mathrm{B},\omega)$, with its source at $\boldsymbol{x}_\mathrm{B}$ anywhere in the half-space below $\mathbb{S}_0$. Note that the superscripts $+$ and $-$ in $f_1^\pm(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega)$ and $G^\pm(\boldsymbol{x},\boldsymbol{x}_\mathrm{B},\omega)$ refer to the propagation direction (downward or upward) at the observation point $\boldsymbol{x}$. The source of the Green's function at $\boldsymbol{x}_{\rm B}$ is omnidirectional. Substituting $q_{\rm A}(\boldsymbol{x},\omega)=0$, $p^{\pm}_{\rm A}(\boldsymbol{x},\omega)=f^{\pm}_1(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)$, $q_{\rm B}(\boldsymbol{x},\omega)=\delta(\boldsymbol{x}-\boldsymbol{x}_{\rm B})$ and $p^{\pm}_{\rm B}(\boldsymbol{x},\omega)=G^{\pm}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)$ into Eqs. (S20) and (S21), using Eqs. (S22) and (S23), gives

$$G^-(\boldsymbol{x}_{\rm A},\boldsymbol{x}_{\rm B},\omega)+\chi(\boldsymbol{x}_{\rm B})f_1(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)$$
$$=\int_{\mathbb{S}_0}\frac{1}{i\omega\rho(\boldsymbol{x})}\Big(\{\partial_iG(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\}f_1(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)-G(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\partial_if_1(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\Big)n_i{\rm d}\boldsymbol{x} \tag{S24}$$

and

$$G^+(\boldsymbol{x}_{\rm A},\boldsymbol{x}_{\rm B},\omega)-\chi(\boldsymbol{x}_{\rm B})f^*_1(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)$$
$$=-\int_{\mathbb{S}_0}\frac{1}{i\omega\rho(\boldsymbol{x})}\Big(\{\partial_iG(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\}f^*_1(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)-G(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\partial_if^*_1(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\Big)n_i{\rm d}\boldsymbol{x}, \tag{S25}$$

respectively, where χ is the characteristic function of the domain $\mathbb{V}_{\rm A}$. It is defined as

$$\chi(\boldsymbol{x}_{\rm B})=\begin{cases}1, & \text{for } \boldsymbol{x}_{\rm B} \text{ between } \mathbb{S}_0 \text{ and } \mathbb{S}_{\rm A},\\ \frac{1}{2}, & \text{for } \boldsymbol{x}_{\rm B} \text{ on } \mathbb{S}=\mathbb{S}_0\cup\mathbb{S}_{\rm A},\\ 0, & \text{for } \boldsymbol{x}_{\rm B} \text{ outside } \mathbb{S}.\end{cases} \tag{S26}$$

Summing Eqs. (S24) and (S25) and using source-receiver reciprocity for the Green's function on the left-hand side yields

$$G(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)+\chi(\boldsymbol{x}_{\rm B})2i\Im\{f_1(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)\}$$
$$=\int_{\mathbb{S}_0}\frac{2}{\omega\rho(\boldsymbol{x})}\Big(\{\partial_iG(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\}\Im\{f_1(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\}-G(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\Im\{\partial_if_1(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\}\Big)n_i{\rm d}\boldsymbol{x}, \tag{S27}$$

where $\Im$ denotes the imaginary part. Taking the real part of both sides of this equation, using Eq. (S16), gives the single-sided representation of the homogeneous Green's function

$$G_{\rm h}(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)=\int_{\mathbb{S}_0}\frac{2}{\omega\rho(\boldsymbol{x})}\Big(\{\partial_iG_{\rm h}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\}\Im\{f_1(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\}-G_{\rm h}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\Im\{\partial_if_1(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\}\Big)n_i{\rm d}\boldsymbol{x}. \tag{S28}$$

S2.3 Single-sided homogeneous Green's function representation: assuming a homogeneous upper half-space

From here onward we assume that also $\mathbb{S}_0$ is a horizontal surface, with $\boldsymbol{n}=(0,0,-1)$. Following a similar derivation as in Appendix B in Wapenaar and Berkhout (1989), we reformulate Eqs. (S18) and (S19) as

$$\int_{\mathbb{V}_{\rm A}}(p^+_{\rm A}q^-_{\rm B}+p^-_{\rm A}q^+_{\rm B}-q^+_{\rm A}p^-_{\rm B}-q^-_{\rm A}p^+_{\rm B}){\rm d}\boldsymbol{x}=$$
$$\int_{\mathbb{S}_0}\frac{2}{i\omega\rho}\Big((\partial_3p^+_{\rm A})p^-_{\rm B}+(\partial_3p^-_{\rm A})p^+_{\rm B}\Big){\rm d}\boldsymbol{x}-\int_{\mathbb{S}_{\rm A}}\frac{2}{i\omega\rho}\Big((\partial_3p^+_{\rm A})p^-_{\rm B}+(\partial_3p^-_{\rm A})p^+_{\rm B}\Big){\rm d}\boldsymbol{x} \tag{S29}$$

and, ignoring evanescent waves,

$$\int_{\mathbb{V}_{\rm A}}(p^{+*}_{\rm A}q^+_{\rm B}+p^{-*}_{\rm A}q^-_{\rm B}+q^{+*}_{\rm A}p^+_{\rm B}+q^{-*}_{\rm A}p^-_{\rm B}){\rm d}\boldsymbol{x}=$$
$$\int_{\mathbb{S}_0}\frac{2}{i\omega\rho}\Big((\partial_3p^+_{\rm A})^*p^+_{\rm B}+(\partial_3p^-_{\rm A})^*p^-_{\rm B}\Big){\rm d}\boldsymbol{x}-\int_{\mathbb{S}_{\rm A}}\frac{2}{i\omega\rho}\Big((\partial_3p^+_{\rm A})^*p^+_{\rm B}+(\partial_3p^-_{\rm A})^*p^-_{\rm B}\Big){\rm d}\boldsymbol{x}. \tag{S30}$$

We apply these theorems to the situation in which the upper half-space above $\mathbb{S}_0$ is homogeneous (for the Green's function as well as for the focusing function). For state A we consider again the focusing function $f_1(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)=f_1^+(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)+f_1^-(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)$, defined in a truncated version of the medium. For state B we consider the Green's function $G(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)=G^{+,+}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)+G^{-,+}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)+G^{+,-}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)+G^{-,-}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)$, with its source at $\boldsymbol{x}_{\rm B}$ anywhere in the half-space below $\mathbb{S}_0$. Note that we introduced two superscripts. The first superscript refers again to the propagation direction at the observation point $\boldsymbol{x}$. The second superscript refers to the radiation direction of the source at $\boldsymbol{x}_{\rm B}$. Substituting $q_{\rm A}^+(\boldsymbol{x},\omega)=q_{\rm A}^-(\boldsymbol{x},\omega)=0$, $p_{\rm A}^\pm(\boldsymbol{x},\omega)=f_1^\pm(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)$, $q_{\rm B}^+(\boldsymbol{x},\omega)=\delta(\boldsymbol{x}-\boldsymbol{x}_{\rm B})$, $q_{\rm B}^-(\boldsymbol{x},\omega)=0$ and $p_{\rm B}^\pm(\boldsymbol{x},\omega)=G^{\pm,+}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)$ into Eqs. (S29) and (S30), using Eqs. (S22) and (S23) and $G^{+,+}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)=0$ for $\boldsymbol{x}$ at $\mathbb{S}_0$ (since the upper half-space is homogeneous), gives

$$G^{-,+}(\boldsymbol{x}_{\rm A},\boldsymbol{x}_{\rm B},\omega)+\chi(\boldsymbol{x}_{\rm B})f_1^-(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)=\int_{\mathbb{S}_0}\frac{2}{i\omega\rho_0}G^{-,+}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\partial_3 f_1^+(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega){\rm d}\boldsymbol{x} \tag{S31}$$

and

$$G^{+,+}(\boldsymbol{x}_{\rm A},\boldsymbol{x}_{\rm B},\omega)-\chi(\boldsymbol{x}_{\rm B})\{f_1^+(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)\}^*=-\int_{\mathbb{S}_0}\frac{2}{i\omega\rho_0}G^{-,+}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\{\partial_3 f_1^-(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\}^*{\rm d}\boldsymbol{x}. \tag{S32}$$

Next, substituting $q_{\rm A}^+(\boldsymbol{x},\omega)=q_{\rm A}^-(\boldsymbol{x},\omega)=0$, $p_{\rm A}^\pm(\boldsymbol{x},\omega)=f_1^\pm(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)$, $q_{\rm B}^+(\boldsymbol{x},\omega)=0$, $q_{\rm B}^-(\boldsymbol{x},\omega)=\delta(\boldsymbol{x}-\boldsymbol{x}_{\rm B})$ and $p_{\rm B}^\pm(\boldsymbol{x},\omega)=G^{\pm,-}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)$ into Eqs. (S29) and (S30), using Eqs. (S22) and (S23) and $G^{+,-}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)=0$ for $\boldsymbol{x}$ at $\mathbb{S}_0$, gives

$$G^{-,-}(\boldsymbol{x}_{\rm A},\boldsymbol{x}_{\rm B},\omega)+\chi(\boldsymbol{x}_{\rm B})f_1^+(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)=\int_{\mathbb{S}_0}\frac{2}{i\omega\rho_0}G^{-,-}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\partial_3 f_1^+(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega){\rm d}\boldsymbol{x} \tag{S33}$$

and

$$G^{+,-}(\boldsymbol{x}_{\rm A},\boldsymbol{x}_{\rm B},\omega)-\chi(\boldsymbol{x}_{\rm B})\{f_1^-(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)\}^*=-\int_{\mathbb{S}_0}\frac{2}{i\omega\rho_0}G^{-,-}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\{\partial_3 f_1^-(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\}^*{\rm d}\boldsymbol{x}. \tag{S34}$$

Summing Eqs. (S31) $-$ (S34), using source-receiver reciprocity for the Green's function on the left-hand side and $G^{+,+}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)=G^{+,-}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)=0$ for $\boldsymbol{x}$ at $\mathbb{S}_0$, we obtain

$$\begin{aligned}&G(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)+\chi(\boldsymbol{x}_{\rm B})2i\Im\{f_1(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)\}\\&\quad=\int_{\mathbb{S}_0}\frac{2}{i\omega\rho_0}G(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\partial_3\bigl(f_1^+(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)-\{f_1^-(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\}^*\bigr){\rm d}\boldsymbol{x}.\end{aligned} \tag{S35}$$

Taking the real part of both sides gives the single-sided representation of the homogeneous Green's function for the situation that the upper half-space is homogeneous

$$G_{\rm h}(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)=4\Re\int_{\mathbb{S}_0}\frac{1}{i\omega\rho_0}G(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)\partial_3\bigl(f_1^+(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)-\{f_1^-(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)\}^*\bigr){\rm d}\boldsymbol{x}. \tag{S36}$$

We conclude by deriving source-receiver reciprocity relations for the decomposed Green's functions $G^{\pm,\pm}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)$. We consider Eq. (S29), but replace $\mathbb{V}_{\rm A}$ by the entire space $\mathbb{R}^3$. In this situation there are only outgoing waves at $\mathbb{S}$. Hence, Eq. (S29) simplifies to

$$\int_{\mathbb{R}^3}\bigl(p_{\rm A}^+q_{\rm B}^-+p_{\rm A}^-q_{\rm B}^+-q_{\rm A}^+p_{\rm B}^--q_{\rm A}^-p_{\rm B}^+\bigr){\rm d}\boldsymbol{x}=0. \tag{S37}$$

First we substitute $q_{\rm A}^+=\delta(\boldsymbol{x}-\boldsymbol{x}_{\rm A})$, $q_{\rm A}^-=0$, $p_{\rm A}^\pm=G^{\pm,+}(\boldsymbol{x},\boldsymbol{x}_{\rm A},\omega)$, $q_{\rm B}^+=\delta(\boldsymbol{x}-\boldsymbol{x}_{\rm B})$, $q_{\rm B}^-=0$ and $p_{\rm B}^\pm=G^{\pm,+}(\boldsymbol{x},\boldsymbol{x}_{\rm B},\omega)$. This gives

$$G^{-,+}(\boldsymbol{x}_{\rm B},\boldsymbol{x}_{\rm A},\omega)=G^{-,+}(\boldsymbol{x}_{\rm A},\boldsymbol{x}_{\rm B},\omega). \tag{S38}$$

Next, we substitute $q_\mathrm{A}^+ = \delta(\boldsymbol{x}-\boldsymbol{x}_\mathrm{A})$, $q_\mathrm{A}^- = 0$, $p_\mathrm{A}^\pm = G^{\pm,+}(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega)$, $q_\mathrm{B}^+ = 0$, $q_\mathrm{B}^- = \delta(\boldsymbol{x}-\boldsymbol{x}_\mathrm{B})$ and $p_\mathrm{B}^\pm = G^{\pm,-}(\boldsymbol{x},\boldsymbol{x}_\mathrm{B},\omega)$. This gives

$$G^{+,+}(\boldsymbol{x}_\mathrm{B},\boldsymbol{x}_\mathrm{A},\omega) = G^{-,-}(\boldsymbol{x}_\mathrm{A},\boldsymbol{x}_\mathrm{B},\omega). \tag{S39}$$

Finally, we substitute $q_\mathrm{A}^+ = 0$, $q_\mathrm{A}^- = \delta(\boldsymbol{x}-\boldsymbol{x}_\mathrm{A})$, $p_\mathrm{A}^\pm = G^{\pm,-}(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega)$, $q_\mathrm{B}^+ = 0$, $q_\mathrm{B}^- = \delta(\boldsymbol{x}-\boldsymbol{x}_\mathrm{B})$ and $p_\mathrm{B}^\pm = G^{\pm,-}(\boldsymbol{x},\boldsymbol{x}_\mathrm{B},\omega)$. This gives

$$G^{+,-}(\boldsymbol{x}_\mathrm{B},\boldsymbol{x}_\mathrm{A},\omega) = G^{+,-}(\boldsymbol{x}_\mathrm{A},\boldsymbol{x}_\mathrm{B},\omega). \tag{S40}$$

Note that Eq. (S39) does not include a minus sign, unlike the corresponding relation for the flux-normalised decomposed Green's functions (Wapenaar, 1996). As a result of this definition, we have the following simple expression for the full Green's function

$$G(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega) = G^{+,+}(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega) + G^{-,+}(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega) + G^{+,-}(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega) + G^{-,-}(\boldsymbol{x},\boldsymbol{x}_\mathrm{A},\omega). \tag{S41}$$

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