A response to the interactive comment on "Regularization methods for the combination of heterogeneous observations using spherical radial basis functions" by Qing Liu et al.

Anonymous Referee #2

Dear Reviewer,

Thank you very much for taking the time to review the manuscript and give comments that help us to improve it. Please find below our point-by-point response to your comments. The original comments are in black, and our answers are in blue. The revised manuscript with tracked changes is also attached.

General comments: The authors present a regularization method that is used in the combination of heterogeneous data. The novelty is the combination of two regularization methods, namely by combining VCE estimation and the L-curve criteria. Various combinations are discussed and compared to existing methods, primarily to VCE estimation or L-curve regularization alone. However, the applied methodology is questionable as the combination of the two criteria is essentially equivalent to a double regularization. The comparison to the calculation "VCE based on CM2" reveals that the same result can be achieved by ordinary approaches. Further, the usage of the Shannon function for analysis and Blackman/CuP function results in additional smoothing which has not been further explained or described. Thus, I consider this paper inconclusive.

We do not apply a double regularization. We combine the VCE for determining the relative weighting between different observation types and the L-curve method for determining the regularization parameter. From the VCE procedure, only the relative weights are kept, the generated regularization parameter is not further used.

The experiments show that 'VCE based on CM 2' cannot guarantee a reliable result. When the Shannon function is used for both analysis and synthesis as suggested, 'VCE based on CM 2' gives even worse results (larger RMS errors as well as smaller correlations) than 'the L-curve method based on CM 1' in the study cases B and D (please refer to Table 6 in the revised manuscript). The 'VCE + L-curve method' overperforms 'VCE based on CM 2' by 6.1%, 48%, 0.7%, 4.1%, 2.3% and 1.8% in terms of RMS error in the six study cases, respectively. When the CuP function is used, 'VCE based on CM 2' gives smaller correlation results than the 'L-curve method based on CM 1' in three study cases. 'VCE + L-curve method' overperforms 'VCE based on CM 2' by 9.7%, 1.1%, 10.5%, 5.4%, 0.9%, 3.8%, respectively. More important, the performance of 'VCE + L-curve method' is stable in all the study cases. Thus, we think the improvement by using our proposed method is in fact significant.

The idea to use certain spherical radial basis function for the analysis and the same or other functions for the synthesis in case of a band limitation is explained by Schreiner (1996), Schmidt et al. (2007, Theorem 1) and Lieb (2017). However, since our goal is to compare different types of regularization parameter choice methods but not different SRBFs, we have changed the experiments to using the same SRBF for both analysis and synthesis. Thus, we have modified Section 2.3, and we have updated the results and discussions in Section 5.3.

Specific comments: Section 1: The motivation of the regularization is unclear. Why are new methods needed? What are the limitations of existing methods? Why is the specific approach of the authors chosen and what benefits do the authors expect from their approach?

We have replaced the content of Page 2, Line 29 to Page 3, Line 9 in Section 1 by the following two paragraphs to explain the motivation of this study in more detail. In the revised manuscript, we explain the limitations of using the L-curve method or the VCE alone, and why we proposed the two new methods.

"VCE estimates the variance components of different observation techniques as well as the regularization parameter simultaneously. However, in this case, the regularization parameter is handled as another variance component, and the prior information is interpreted as an additional observation technique, and, thus, assumed to be of random character. In most of the regional gravity modeling studies, a background model serves as prior information. In this case, the prior information has no random character, and the regularization parameter generated by VCE is not reliable (Liang, 2017). Lieb (2017, p.131) presents a case which shows the instability of VCE. Naeimi (2013, p.102) shows that VCE generally performs worse than the L-curve method.

Since VCE does not guarantee a reliable regularization solution, and the L-curve method cannot weight heterogeneous observations, the purpose of this paper is to combine these two methods, and to improve the stability and reliability of the solutions. The idea of combining VCE for weighting different data sets and a method for determining the regularization parameter was introduced in the Section 'future work' of both Naeimi (2013, p.121) and Liang (2017, p.134). The study in this manuscript is also inspired by Wang et al. (2018), who combine two methods successively for determining the regularization parameter and relative weights for GPS and InSAR. However, to the best of our knowledge, there are still no publications applying this idea for combining heterogeneous observations in regional gravity field modeling. Thus, we introduce and discuss in the paper the two proposed new methods which combine VCE for determining the relative weighting between different observation types and the L-curve method for determining the regularization parameter, denoted as 'VCE + L-curve method' and 'L-curve method + VCE', depending on the order of the applied procedures. Numerical experiments are carried out to compare their performance to the original L-curve method and VCE."

page 3, line 5: The authors argue to find the best-performing method (in what sense?) for regularization. However, they do not consider other methods than VCE and L-curve, e.g. GCV. Further, the method will be best-performing for their specific problem as no general criteria is derived which allows to conclude that the proposed method is best-performing.

We agree that the word “best-performing” is too “big” for this paper. The purpose of this paper is to test if the two combined methods give better results compared to VCE or the L-curve method alone. Since the L-curve method or other conventional regularization parameter choice methods cannot weight heterogeneous observations, and VCE does not guarantee a reliable regularization solution, we want to improve the stability and reliability of the solutions by combining these two methods. The criteria used for comparing the performance are the RMS error as well as the correlation between the estimated coefficients and the validation data (see Section 5.2, Page 11, Line 20-29).
We have already changed the whole paragraph corresponding to the first specific comment and have rewritten the purpose of the paper. Please refer to the answer of the last comment and the revised manuscript.

Section 2.3: The authors present three different SRBFs with various smoothing features. Why is the approach of Eicker (2008) not considered? By including gravity field information into Bn, a considerable improvement can be achieved.

The focus of this work is not to compare the performance of different SRBFs, but to compare the performance of different regularization parameter choice methods. For each group of comparison, the same SRBF is used for every regularization parameter choice method. However, as mentioned in the future work, we plan to study the performance of more types of SRBFs using the newly devised method.

Section 2.3: If I understood the author’s approach correctly, they use the Shannon function for the analysis of the simulated data but apply the estimated coefficients using either the Blackman or CuP function in the synthesis step. This approach is at least odd and inconsistent if not wrong from the beginning. In-fact, the approach introduces an additional smoothing. The authors state correctly that the latter two have smoothing features. Thus, the approach is unsuitable for the conducted research as it masks the effects of the regularization. It is another implicit regularization and thus the results cannot unambiguously assigned to the performance of the chosen methods. The only correct approach is therefore to use the same function for the analysis and synthesis step. The approach is even more questionable as Bentel2013 showed that differences between SRBFs matter (as also stated by the authors).

In the previous version of the manuscript, we have conducted two sets of experiments; the first set uses the Shannon function for analysis and the Blackman function for synthesis; the second set uses both the CuP function for analysis and synthesis.
Schreiner (1996) and Freeden et al. (1998) gave the proof that different types of SRBFs can be used in the analysis step and synthesis step in case of the same band limitation.

This procedure was applied in Lieb et al. (2016), Lieb (2017), among many others. However, since our goal is to compare different types of regularization parameter choice methods but not different SRBFs, we have changed the first set of experiments to use the Shannon function for both analysis and synthesis. Thus, we have removed the Blackman function and modified Section 2.3, consequently, we have also updated the RMS results for the first set of experiments and the corresponding discussions in Section 5.3.

Section 3.1 provides no new information. The content can be reduced to the most significant equations and appropriate referencing.

We have shortened some of the content. Since Section 3.1 is not very long, and it discusses how the coefficients are estimated and how the regularization parameter is introduced, we have kept significant equations.

Section 3.2: CM1 can obviously be removed as the assumption \sigma_1ˆ2 = \sigma_2ˆ2 = ... is hardly valid in any case (except for simulated data with exactly this assumption). Furthermore, applying VCE is the proper tool to consider data with varying variance factors. Thus, the results of CM1 are superfluous and the results prove the invalidity of the assumption.

The ordinary L-curve method can only be applied based on CM 1 because it cannot estimate the variance factors. And the results show that although results based on CM 1 are expected to be worse than those based on CM 2, ‘the L-curve method based on CM 1’ still performs better than ‘VCE based on CM 2’ in some study cases. So, the results of CM 1 prove that VCE does not guarantee reliable regularization results, and thus show the importance of our combined method.
However, we have removed the method ‘VCE based on CM 1’ since the variance factors need to be considered for different data sets, and the results from ‘VCE based on CM 1’ are expected and proved to be worse than ‘VCE based on CM 2’. Section 5.3 and 6 are thus rewritten.

Section 4.3: The regularization is essentially a double differentiation as the estimated variance factors during the VCE will reflect the regularization parameters. Practically the \lambda of equation 30 is split in \lambda_1 + \lambda_2 where one is estimated by VCE and the other by the L-curve criterion or vice-versa. Due to the double regularization, the results will be further smoothed than in case of applying just one of the methods alone. A better fit is therefore expected as the inherent effects due to ill-posedness is dominating. Also, the authors do not motivate the need for a second regularization and also do not discuss the effect of the second regularization step.

We do not apply a double regularization. We used VCE for determining the relative weight between each observation types and the L-curve method for determining the regularization parameter. The regularization parameter that was generated from VCE is

not further used. The λ of Eq. (30) is only estimated by the L-curve criterion, and the relative weights ω_p in Eq. (30) are estimated by VCE.
For clarification, we have extended the description part in Section 4.3 in the revised paper.

Section 5.3: The authors present two study cases: A and F; why not naming them A and B as you only present results of those two. The reader will have no information on cases B to E. Further, the results of CuP function can also be removed as they do not introduce any new insight.

The naming depends on how many types of observations are combined; from A to F, more types of observations are combined. However, we have rewritten Section 5.3, because we have changed the first set of experiments to using the Shannon function for both analysis and synthesis, and the results from 'VCE based on CM 1' are removed. In the revised version, we discuss all the cases together, and Section 5.3 is rewritten.
We think the results of the CuP function are necessary for two reasons. The first reason is that Naeimi (2013, p. 121) points out that VCE gives better performance when smoothing kernels which have built-in regularization are used. Our results of the CuP function show that even with a built-in regularization, VCE still does not guarantee a reliable result, and 'VCE + L-curve method' outperforms VCE in all the study cases. The second reason is that when the Shannon function is used for both analysis and synthesis, 'VCE + L-curve method' always outperforms the original L-curve method and VCE, and 'L-curve method + VCE' also generally outperforms the L-curve method and VCE. But when the CuP function is used, 'VCE + L-curve method' still performs the best but 'L-curve method + VCE' does not show significant improvements compared to VCE. So, we conclude that the 'VCE + L-curve method' improves the stability and reliability of the solution no matter the used SRBFs have a smoothing feature or not.

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From 21 Jun 2019 to 28 Nov 2019 a revised version of the manuscript including track changes was available in this supplement. Upon the authors' request it was removed.