My comments are organized following the paragraph of the paper

I discover the level-set method with this paper and I had a hard time understanding it from the manuscript. I had then to go through the reading of several papers before entering the manuscript.
On one hand, it's ok, these reading are necessary for learning this method from scratch. On the other hand, the **method** paragraph of the manuscript turned to be of no help to understand it.

2.1 **Generalized level-set method**

I thus had two readings of the summary of the method. I) Reading as a novice. (that I was), in that case this part is just totally incomprehensible; ii) reading as an expert (that I'm almost now…), the part is still confusing and do not contain the important information. In both cases I felt quite frustrated.

The authors have the choice, i) either they consider that Giraud et al. 2021 paper (referred below as G21) is a mandatory reading, and then remove the method summary, or ii) they give the reader enough material to keep reading the paper before reading G21, if necessary. I think the second solution is the correct one, and without increasing the size, then can give a clear, synthetic description of the problem settings.

I suggest relying on figure 1 that is quite clear and replacing the present method part by:

- Starting with a geometrical description: medium is discretized by cell/nodes (unclear); model is defined by different geological/geophysical units with boundaries (defined on the same mesh); properties are kept constant in geological units; Hence, define N, M, the scope of phi_k

- Boundaries; recall in few words and/or reference level-set and signed distance. A simple drawing showing a 1D phi_k across a boundary with a true versus "smeared" Heaviside would help. Explain what the authors means by a "multinary structure" or leave that to a reading of G21. I thing that eq. 7 and/or 8 of G21 is worth being recalled here.

- Setting of the inverse problem.
  Eq 1 alone can be misleading. It is worth recalling that it comes from a linearization of the problem. I didn't find the information about the iterative scheme that is used to solve the non-linear problem, I guess it's a steepest descent.

Below are some remarks about the text:

1) You use throughout the text the notion of "rock unit". It seems to me that "Geological or geophysical unit" would more appropriate since you can deal also with sand, clay, salt, etc.

   line 83: you introduce signed-distance values to interface calculated by FMM. Without further explanation, this sentence is totally incomprehensible. Outline is inappropriate, use boundary or interface instead.

2) Ligne 85: the sentence where you transform a "signed distance" to a multinary structure (???) using a smeared-out Heaviside is obscure.

Line 91-102: This paragraph is very confusing and for me incorrect.
The sentence “Initializing the model space…” is confusing. **m()** is the model function that links the modeled data to the parameters, through the signed distance Φ_k. It is not a space, neither in a mathematical sense nor in geometrical sense. And you do not “initialize” a model, unless you talk about the initial (trial) model, you “define” it. You'd rather stick to G21 formulation in this part.

Eq 1 is totally confusing since it mixes a general and an iterative formulation. What is d^{calc}? It is never defined.
I suggest to rewrite this paragraph according to a more standard way of presenting inverse problems:

a) You are interested in solving a discrete inverse problem whose direct formulation is:

d=**g**(**m**); **d**= data; **m** parameter to be inverted; **g**() the direct function, non linear in our case.

b) You decide to solve this non linear problem using a gradient type method base on a 1st order Taylor expansion

$$\mathbf{g}(\mathbf{m}) \approx \mathbf{g}(\mathbf{m}_0) + \left.\frac{\partial \mathbf{g}}{\partial \mathbf{m}}\right|_{\mathbf{m_0}} (m - m_0)$$

c) Considering the parameters of your direct problem: $\mathbf{m} = \mathbf{m}(\Phi, \rho)$ in which density is kept constant, this turns into:

$$\mathbf{g}(\mathbf{m}) \approx \mathbf{g}(\Phi_0) + \left.\frac{\partial \mathbf{g}}{\partial \mathbf{m}} \frac{\partial \mathbf{m}}{\partial \Phi}\right|_{\Phi_0} (\Phi - \Phi_0) \Leftrightarrow \mathbf{g}(\mathbf{m}(\Phi)) = \mathbf{g}(\Phi_0) + \mathbf{J}^{\Phi}\delta\Phi$$

d) And you decide to iteratively minimize in a least square sense:

$\Psi^r_{i+1} = \left\| \mathbf{d}^{obs} - \mathbf{g}(\Phi_i) - \mathbf{J}^{\Phi_i}\delta\Phi \right\|_2$ where now $\mathbf{d}^{calc} = \mathbf{g}(\Phi i)$ is defined as the result of the direct problem at iteration i.

Please note that compared to your eq1, I have a sign difference. You never use the residuals r that is defined in line94, is it necessary?

2.2 Regularization level-set inversion

Sentences in lines 118-121 are confusing and the statement is incorrect, this regularization does not “*encourage the $\delta\Phi$ update to reach specific values stored in **q***”, but it does “*encourage the product **W**$\delta\Phi$ update to reach specific values stored in **q***” which is quite different (imagine that W is a Laplacian, or a smoothing operator). Since at this point neither **W** nor **q** are defined, it is difficult to understand what the authors mean.
I suggest that the authors replace the text that is too general by more precise details that are given later in the text.
What is the exact size of **q** vector?

Do you try to impose something like $\left\| \begin{matrix} \mathbf{W}_S\delta\Phi \\ \mathbf{W}_P\delta\Phi - \mathbf{v} \end{matrix} \right\|$ minimum?

Besides, why do you mix these two constraints simultaneously?

What is the difference between imposing eq 4 rather than $\left\|\mathbf{W}_S \delta\Phi\right\| + \left\|\mathbf{W}_P \delta\Phi - \mathbf{v}\right\|$ minimum? or $\left(\delta\Phi - \delta\Phi_{prior}\right)^T C_\Phi^{-1}\left(\delta\Phi - \delta\Phi_{prior}\right)$?

Should we interpret W_p as a geometrical mask (rather than a weighting) that allows fixing some specific values of boundaries in the different geological units?

2.3

The sentence on line 149 is incomprehensible, and the full paragraph from 148-153 confusing.

2.4 Your explanations are ok, however it is difficult to grasp the influence of this topological rule enforcement on your results. Could you comment on the effects of this processing on the synthetic case for instance?

3.

Figure 5 caption: what do you mean by difference between "new data and synthetic", what are the new data?

A general question: in 4.3 and 4.4 you choose to build a starting model from the inversion of density only, then invert for the interfaces only in a second step. Why don't you try to invert simultaneously for interfaces and density values in the different units?

Line 384: "due to … sections". Use a direct formulation instead: "We present … because…"

4.5.2

I do not understand which geometrical constraints you apply from the seismic profile. On figure 2d for the synthetic case, we clearly see that your constraint follows the geometry of the reflector. What about results obtained on figure 12? There are no clear reflectors such as those of synthetic examples, but rather several general eastward dipping trends. Which constraints do you apply? Could you provide a plot of these constraints along the 2D section?

You mention on line 459 that Goleby et al. (2004) and Lindsay et al.(2020) use different seismic profiles. On line 341 you mention that you use Goleby interpretation. Is your seismic profile coming from the 2004 study or the 2020 study?